ferences in $(k\rho c)^{1/2}$ appear to explain, at least in part, the discrepancies attributed to the thermal paint technique.

Figure 2 presents similar data for a W-66 stycast resin containing a copper powder filler with a large concentration gradient between the upper and lower surfaces of the sample. Whereas copper normally is not used as a filler for model fabrication, it does represent an extreme case of inhomogeneity and therefore a formidable test of the technique. Because of the material inhomogeneity, the effective curve now becomes a function of either heating rate or time as well as surface temperature. Here again, there is a significant difference between the measured steady-state thermophysical property and the effective value, particularly at the initial temperature. The use of measured steady-state properties, which are averaged over the sample thickness and are functions of overall temperature only, rather than effective properties, again would introduce significant errors into the calculation of the aerodynamic heat-transfer coefficient. In fact, nearly all model materials showed significant differences between the measured steady-state and effective properties.

Conclusions

This paper has shown that an effective thermophysical property may improve significantly the accuracy of the phase-change coating technique in estimating local heat-transfer coefficients. The apparatus described in Ref. 5 permits direct measurement of $(k\rho c)^{1/2}$ for different heating levels for models under this study. Additionally, preliminary work at Langley Research Center has shown that dynamic thermophysical property measurements can be obtained significantly faster than the steady-state techniques currently used. In light of the potential improvement in accuracy, and increased speed, the use of an effective thermophysical property to reduce phase-change paint data is recommended.

References

[1] Jones, R. A. and Hunt, J. L. "Use of Fusible Temperature Indicators for Obtaining Quantitative Aerodynamic Heat Transfer Data," TR R-230, Feb. 1966, NASA.

[2] Drummond, J. P., "A Method for Improving the Accuracy in Phase Change Heat Transfer Data Through Increased Precision in Thermophysical Property Determinations," M. E. Thesis, April 1975, Old Dominion Univ., Norfolk, Va.

[3] Hunt, J. L., Pitts, J. I., and Richie, C. B., "Application of Phase Change Technique to Thin Sections with Heating on Both Sides," TN-D-7193, Aug. 1973, NASA.

[4] Maise, G., and Rossi, M. J., "Lateral Conduction Effects on Heat Transfer Data Obtained with the Phase Change Paint Technique," CR-2435, Aug. 1974, NASA.

[5] Corwin, R. R. and Kramer, J. S., "An Electronic System for Measuring Thermophysical Properties of Wind Tunnel Models," CR-2511, March 1975, NASA.

[6] Myers, G. E., *Analytical Methods in Conduction Heat Transfer*, McGraw-Hill, N. Y. 1971.

[7] Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids*, Oxford University Press, 1959.

The Response of Infinite Thin Shells to Initial Stress

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I. Introduction

SUPPOSE we have a long thin cylindrical shell which, although in the past may have been deforming plastically, is currently vibrating elastically. If this shell is deforming axisymmetrically then it is a fairly simple task to decide from an experimental strain trace what the equilibrium position would be if the experiment were continued long enough to allow the damping forces to reduce the amplitude of the vibration to insignificant values. In cases where the motion is asymmetric, however, is not quite so simple and may present a formidable analysis task if the equilibrium state is to be derived from direct observation.

The object of this paper is to compute the static equilibrium stress state of an elastically vibrating long thin cylindrical shell from the known stresses at any instant, for the general asymmetric case. Because of its previous history of plastic deformation this final equilibrium state will not be stress free. However, although the initial stress state varies with angle around the cylinder and the equilibrium strain state may therefore be quite a complex function of angle, it will be shown that the equilibrium stress must reduce to a very simple form.

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II. Basic Equations

For an infinite cylindrical thin shell in a state of plane strain, the hoop mean and curvature strains are given by

$$\xi_\phi = \frac{1}{a}\frac{\partial v}{\partial \phi} - \frac{w}{a} \tag{1}$$

$$\gamma_\phi = \frac{1}{a^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) \tag{2}$$

where a is the cylinder radius and v and w the circumferential and radial (+ve inward) displacements from the initially undeformed ring. The mean and moment of the hoop stress are given by

$$N_\phi = \frac{Et}{(1-\nu^2)}\xi_\phi + N_\phi' \tag{3}$$

$$M_\phi = \frac{-Et^3}{12(1-\nu^2)}\gamma_\phi + M_\phi' \tag{4}$$

where N_ϕ, M_ϕ is the initial stress state, E is Young's modulus, ν Poisson's ratio, and t the shell thickness.

From Timoshenko[1] the full equilibrium equations for a thin cylindrical shell are

$$\frac{1}{a^2}\frac{\partial^2 v}{\partial \phi^2} - \frac{1}{a^2}\frac{\partial w}{\partial \phi} + \frac{t^2}{12a^4}\frac{\partial^3 w}{\partial \phi^3} + \frac{t^2}{12a^4}\frac{\partial^2 v}{\partial \phi^2} = -\left(\frac{1-\nu^2}{Et}\right)\left(\frac{1}{a}\frac{\partial N_\phi'}{\partial \phi} - \frac{1}{a^2}\frac{\partial M_\phi'}{\partial \phi}\right) \tag{5}$$

$$\frac{1}{a^2}\frac{\partial v}{\partial \phi} - \frac{w}{a^2} - \frac{t^2}{12a^4}\frac{\partial^4 w}{\partial \phi^4} - \frac{t^2}{12a^4}\frac{\partial^3 v}{\partial \phi^3} = -\left(\frac{1-\nu^2}{Et}\right)\left(\frac{N_\phi'}{a} + \frac{1}{a^2}\frac{\partial^2 M_\phi'}{\partial \phi^2}\right) \tag{6}$$

The initial stress state N_ϕ', M_ϕ' may be a general function of ϕ but there is no loss of generality in assuming that each may be expanded as a Fourier series, i.e.

$$N_\phi' = N_o + \sum_{n=1,2...} N_n \cos n\phi \tag{7}$$

$$M_\phi' = M_o + \sum_{n=1,2...} M_n \cos n\phi \tag{8}$$

Our final equilibrium stress state may be found by superposing solutions for each of the N_i and M_i.

In the first instance let us examine the equilibrium position for an initial stress independent of ϕ, i.e. (N_0, M_0). From symmetry considerations we have $v=0$. Then Eq. (5) is satisfied identically and Eq. (6) simplifies to

$$w=\frac{1-v^2}{Et}aN_o: \tag{9}$$

appropriate substitution gives

$$\xi_\phi=-\frac{1-v^2}{Et}N_o \tag{10a}$$

$$\gamma_\phi=0 \tag{10b}$$

and equilibrium stress $N_\phi=0$, $M_\phi=M_0$, i.e., the residual mean force is zero and the residual moment is equal to a constant.

Now consider the more complicated case when

$$N'_\phi=N_n\cos n\phi \tag{11a}$$

$$M'_\phi=M_n\cos n\phi \tag{11b}$$

A simple examination of the equilibrium Eqs. (5) and (6) shows that the solutions for v and w are of the form

$$v=v_n\sin n\phi \tag{12a}$$

$$w=w_n\cos n\phi \tag{12b}$$

and substituting in Eqs. (5) and (6) gives

$$-\frac{n^2}{a^2}\left(1+\frac{t^2}{12a^2}\right)v_n+\frac{n}{a^2}\left(1+\frac{t^2n^2}{12a^2}\right)w_n=G_2 \tag{13}$$

$$\frac{n}{a^2}\left(1+\frac{t^2n^2}{12a^2}\right)v_n-\frac{1}{a^2}\left(1+\frac{t^2n^4}{12a^2}\right)w_n=G_3 \tag{14}$$

where

$$G_2=\frac{1-v^2}{Et}\frac{1}{a}(nN_n-nM_n/a)$$

and

$$G_3=\frac{1-v^2}{Et}\frac{1}{a}(-N_n+n^2M_n/a)$$

The solution to these equations for $n\neq 1$ is

$$v_n=\frac{-12a^4}{t^2n^2(n^2-1)^2}(nG_3+G_2)-\frac{a^2n^2}{(n^2-1)^2}(G_2+G_3/n) \tag{15}$$

$$w_n=\frac{-12a^4}{t^2n(n^2-1)^2}(nG_3+G_2)-\frac{a^2n}{(n^2-1)^2}(G_2+G_3/n) \tag{16}$$

and if we write

$$\beta_n=12(1-v^2)a^2M_n/Et^3 \tag{17}$$

and

$$\alpha_n=a(1-v^2)N_n/Et \tag{18}$$

we obtain

$$v=(-\beta_n/n-n\alpha_n)\sin n\phi/(n^2-1) \tag{19a}$$

$$w=(-\beta_n-\alpha_n)\cos n\phi/(n^2-1) \tag{19b}$$

Completing the solution we have, for each $n\geq 2$

$$\xi_\phi=-(\alpha_n/a)\cos n\phi \tag{19c}$$

$$\gamma_\phi=(\beta_n/a^2)\cos n\phi \tag{19d}$$

$$N_\phi=0 \tag{19e}$$

$$M_\phi=0 \tag{19f}$$

When $n=1$ the determinant of Eqs. (13) and (14) vanishes implying no unique solution. This is understandable as the motion $v=d\sin\phi$, $w=d\cos\phi$ is a rigid body displacement by an amount d. Accordingly for the first harmonic we have w, arbitrary and

$$v=w_1-\alpha_1(1-M_1/aN_1)/(1+t^2/12a^2)\}\sin\phi \tag{20a}$$

$$w=w_1\cos\phi \tag{20b}$$

$$\xi_\phi=-\{\alpha_1(1-M_1/aN_1)/a(1+t^2/12a^2)\}\cos\phi \tag{20c}$$

$$\gamma_\phi=-\{\alpha_1(1-M_1/aN_1)/a^2(1+t^2/12a^2)\}\cos\phi \tag{20d}$$

$$N_\phi=\{-(aN_1-M_1)/a(1+t^2/12a^2)+N_1\}\cos\phi \tag{20e}$$

$$M_\phi=\{t^2(aN_1-M_1)/12a^2(1+t^2/12a^2)+M_1\}\cos\phi \tag{20f}$$

Neglecting terms of order $(t/a)^2$ gives

$$v=\{w_1-\alpha_1(1-M_1(aN_1)\}\sin\phi \tag{21a}$$

$$w=w_1\cos\phi \tag{21b}$$

$$\xi_\phi=-\{\alpha_1(1-M_1/aN_1)/a\}\cos\phi \tag{21c}$$

$$\gamma_\phi=-\{\alpha_1(1-M_1/aN_1)/a^2\}\cos\phi \tag{21d}$$

$$N_\phi=\{(M_1+N_1t^2/12a)/a\}\cos\phi \tag{21e}$$

$$M_\phi=\{M_1+N_1t^2/12a\}\cos\phi \tag{21f}$$

It is interesting to note that N_ϕ and M_ϕ are only zero implicitly for $n\geq 2$.

It can be seen from Eqs. (19) and (21) that the final solution is given by the low-order harmonics and consequently it is not sufficiently accurate to use the Donnell form of the equations where some terms of low order on the left-hand side of Eqs. (5) and (6) are neglected. A fuller description of this is given by Dym[2] but it may be noted that the Donnell form of the equations does not lead to the correct rigid body displacement solutions for this problem.

Combining the solutions derived for $n=0$, 1 and ≥ 2 we have a complete solution to the equilibrium state for a thin cylindrical shell with an initial stress state given by N'_ϕ and M'_ϕ, as follows

$$v=d\sin\phi-\alpha_1(1-M_1/aN_1)\sin\phi-\sum_{n=2,3}\{(\beta_n/n+n\alpha_n)/(n^2-1)\}\sin n\phi \tag{22a}$$

$$w=\alpha_0+d\cos\phi-\sum_{n=2,3...}\{(\beta_n+\alpha_n)/(n^2-1)\}\cos n\phi \tag{22b}$$

$$\xi_\phi=-\alpha_o/a-\{\alpha_1(1-M_1/aN_1)/a\}\cos\phi-\sum_{n=2,3...}(\alpha_n/a)\cos n\phi \tag{22c}$$

$$\gamma_\phi=-\{\alpha_1(1-M_1/aN_1)/a^2\}\cos\phi+\sum_{n=2,3...}(\beta_n/a^2)\cos n\phi \tag{22d}$$

$$N_\phi = \{ (M_1 + N_1 t^2/12a)/a \} \cos\phi \tag{23a}$$

$$M_\phi = M_0 + (M_1 + N_1 t^2/12a) \cos\phi \tag{23b}$$

III. Discussion

From the harmonic analysis of the previous section it can be seen that for any value of n there is a nonzero value for the equilibrium strain which, following superposition, will lead to quite a complex function for the general case. For the stress, however, it has been shown that the equilibrium state for a harmonic greater than or equal to two is zero. Hence for any initial stress N'_ϕ, M'_ϕ the equilibrium stress is as given by Eq. (23) where M_0, N_1, and M_1 are, respectively, the mean moment and first harmonics of the initial stress N'_ϕ, M'_ϕ.

References

[1]Timoshenko, S. P., *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[2]Dym, C. L., *Introduction to the Theory of Shells*, Pergamon Press, New York, 1974.

Timoshenko's Conjecture on Buckling of Annular Plates under Uniform External Pressure

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I. Introduction

ABOUT four decades ago Timoshenko,[1] conjectured while discussing Meissner's[2] results for buckling of annular plates under external pressure that for large holes the plate with clamped outer and free inner edges buckles in many circumferential waves. His comment was based on the physical argument that the conditions in such a plate (ring) are analogous to those of a long compressed rectangular plate clamped along one side and free along the other.

The previous conjecture was verified much later through numerical results by Rozsa[3] and recently by other investigators.[4-7] In Refs. 3 and 4, certain correlations which are different, one from the other, have been proposed between the physical parameters of the annulus and those of the infinite strip.

The object of the present Note is to offer a mathematical explanation for Timoshenko's conjecture and extend it to various combinations of clamped, simply supported, and free edge conditions. In addition, new correlations through a limiting process are derived. These are found to be useful in obtaining, a priori, a starting value for the number of circumferential waves in the critical buckling mode of a given annulus.

II. Analysis

A thin annular plate of uniform thickness h under external pressure p_0 is considered (see Fig. 1a), and the material of the plate is assumed to be homogeneous and isotropic. The prebuckling membrane stresses in such a plate are given by the well-known Lame's solutions. For a plate buckling in n circumferential waves, the lateral deflection $W(r,\theta)$ is represented in the from

$$W = W_n(r) \cos(n\theta + \epsilon) \tag{1}$$

Then the strain energy V of bending, and the work done T by the mid-plane forces during bending assume the forms

$$V = \frac{\pi}{2} D \int_a^b \Big[\Big(\frac{d^2 W_n}{dr^2} + \frac{1}{r}\frac{dW_n}{dr} - \frac{n^2}{r^2} W_n \Big)^2 - 2(1-\nu)\frac{d^2 W_n}{dr^2}\Big(\frac{1}{r}\frac{dW_n}{dr} - \frac{n^2}{r^2}W_n\Big) + 2(1-\nu)\frac{n^2}{r^2}\Big(\frac{dW_n}{dr} - \frac{W_n}{r}\Big)^2 \Big] r\,dr \tag{2}$$

and

$$T = \frac{\pi p_0 b^2 h}{2 b^2 - a^2} \Big[\int_a^b \Big(1 - \frac{a^2}{r^2}\Big)\Big(\frac{dW_n}{dr}\Big)^2 r\,dr + \int_a^b \Big(1 + \frac{a^2}{r^2}\Big)\Big(\frac{n}{r}W_n\Big)^2 r\,dr \Big] \tag{3}$$

in which ν is the Poisson's ratio, and $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate. The eigenmodes and eigenloads are governed by the variational problem $\delta(V-T)=0$ with respect to arbitrary variation δW_n satisfying relevant geometric boundary conditions.

The above-mentioned variational problem is expressed in more convenient form by using the transformations[6]

$$W_n(r) = r^2 w(r) \quad \text{and} \quad y = (r-a)/(b-a) \tag{4}$$

in succession. The integrand in the problem thus modified is found to be a fifth-order polynominal in the parameter $(b-a)/a$ with an eigenvalue parameter properly defined. In the limit $a \mapsto b$, the problem reduces to

$$\delta \Big[\int_0^1 \Big(\frac{1}{\beta}\frac{d^2 w}{dy^2} - \beta w \Big)^2 dy + 2(1-\nu)\int_0^1 \Big\{ W\frac{d^2 w}{dy^2} + \Big(\frac{dw}{dy}\Big)^2 \Big\} dy - \lambda \int_0^1 w^2 dy \Big] = 0 \tag{5}$$

in which

$$\beta = n(b-a)/a$$

and

$$\lambda = 2(p_0 b^2 h/D)(b-a)/(b+a)$$

The above variational problem Eq. (5) is identical to that of the buckling of a long rectangular strip of length a' and width b' if one takes $\lambda = \sigma h b'^2/D$ and $\beta = m\pi b'/a'$ where σ is compressive stress at the shorter edges, and m is number of half-waves along the length of the strip (see Fig. 1b). By taking $b' = (b-a)$, as in Refs. 3 and 4, the correlation between σ and p_0 becomes

$$\sigma = 2p_0 b^2/(b^2 - a^2) \tag{6}$$

which is equal to the compressive hoop stress at the inner edge of the annulus. It may be mentioned here that σ in the corresponding correlation in Ref. 3 is equal to the average hoop stress along radial edge, and in Ref. 4, it is equal to the hoop stress at the outer edge.

The least eigenvalue λ for various values of β are obtained by the Rayleigh-Ritz method with simple polynomials in y as

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